TOPOLOGIA E GEOMETRIA DIFFERENZIALE

a.a. 2014/15

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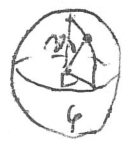
(1) Sia data, in $\mathbb{R}^3 \setminus \{(0,0,0)\}$,

$$E = \frac{1}{(x^2+y^2+z^2)^{3/2}} \left(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \right)$$

Si consideri $f : \mathbb{R}^3 \circlearrowleft$ $\quad \begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\vartheta \end{cases}$ $\quad \rho \geq 0$, $\varphi \in [0, 2\pi)$, $\vartheta \in [0, \pi]$

(coord. sferiche)

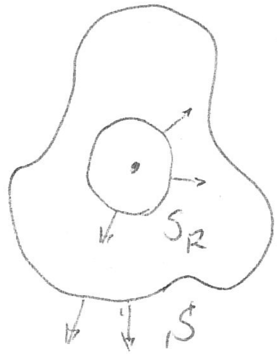


Determinare f^*E; verificare che E è chiusa

(2) Dimostrare che

$$\int_S E = \int_{S_R} E \qquad \forall R > 0$$

(v. figura)



(si lavori con $f^*E \ldots$)

(3) E è esatta? Spiegare

Tempo a disposizione: 1h.

Le risposte vanno adeguatamente giustificate

1
2
3

$\mathbb{R}^3 \setminus \{(0,0,0)\}$

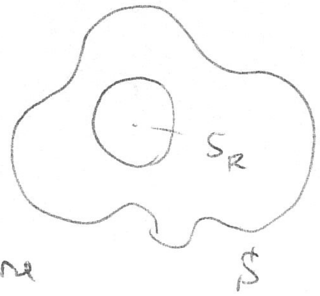
$$E = \frac{1}{(x^2+y^2+z^2)^{3/2}} \left(x\, dy\wedge dz + y\, dz\wedge dx + z\, dx \wedge dy \right)$$

$$f : \mathbb{R}^3 \circlearrowleft \qquad \begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\vartheta \end{cases} \qquad \begin{array}{l} \rho \geq 0 \\ \varphi \in [0, 2\pi) \\ \vartheta \in [0, \pi] \end{array}$$

a) determinare f^*E; dimostrare che E è chiusa

b) Dimostrare che

$$\int_S E = \int_{S_R} E \qquad \forall R > 0$$



c) E, f^*E sono esatte? Spiegare

Sol.

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(\rho,\vartheta)}\, d\rho\wedge d\vartheta + \frac{\partial(y,z)}{\partial(\vartheta,\varphi)}\, d\vartheta \wedge d\varphi + \frac{\partial(y,z)}{\partial(\varphi,\rho)}\, d\varphi\wedge d\rho$$

$$\frac{\partial(y,z)}{\partial(\rho,\vartheta)} = \begin{vmatrix} \sin\vartheta\sin\varphi & \rho\cos\vartheta\sin\varphi \\ \cos\vartheta & -\rho\sin\vartheta \end{vmatrix} = -\rho\sin^2\vartheta\sin\varphi - \rho\cos^2\vartheta\sin\varphi$$

$$= -\rho\sin\varphi$$

$$\parallel$$

$$\begin{vmatrix} \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \vartheta} \end{vmatrix}$$

$$\frac{\partial(y,z)}{\partial(\vartheta,\varphi)} = \begin{vmatrix} \rho\cos\vartheta\sin\varphi & \rho\sin\vartheta\cos\varphi \\ -\rho\sin\vartheta & 0 \end{vmatrix} = +\rho^2\sin^2\vartheta\cos\varphi$$

$$\frac{\partial(y,z)}{\partial(\varphi,\rho)} = \begin{vmatrix} \rho\sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi \\ 0 & \cos\vartheta \end{vmatrix} = \rho\sin\vartheta\cos\vartheta\cos\varphi$$

$$\left[x\, dy\wedge dz = \right. \quad \rho \sin\vartheta \cos\varphi \Big[-\rho \sin\varphi \, d\rho\wedge d\vartheta + \rho^2 \sin^2\vartheta \cos\varphi \, d\vartheta\wedge d\varphi + \rho \sin\vartheta \cos\vartheta \cos\varphi \, d\varphi\wedge d\rho \Big]$$

$$= -\rho^2 \sin\vartheta \cos\varphi \sin\varphi \; d\rho\wedge d\vartheta + \rho^3 \sin^3\vartheta \cos^2\varphi \; d\vartheta\wedge d\varphi + \rho^2 \sin^2\vartheta \cos\vartheta \cos^2\varphi \; d\varphi\wedge d\rho \Big]$$

$$dz\wedge dx = \frac{\partial(z,x)}{\partial(\rho,\vartheta)} d\rho\wedge d\vartheta + \frac{\partial(z,x)}{\partial(\vartheta,\varphi)} d\vartheta\wedge d\varphi + \frac{\partial(z,x)}{\partial(\varphi,\rho)} d\varphi\wedge d\rho$$

$$\frac{\partial(z,x)}{\partial(\rho,\vartheta)} = \begin{vmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta\cos\varphi & \rho\cos\vartheta\cos\varphi \end{vmatrix} = \rho\cos^2\vartheta\cos\varphi + \rho\sin^2\vartheta\cos\varphi = \rho\cos\varphi$$

$$\frac{\partial(z,x)}{\partial(\vartheta,\varphi)} = \begin{vmatrix} -\rho\sin\vartheta & 0 \\ \bullet & -\rho\sin\vartheta\sin\varphi \end{vmatrix} = \rho^2\sin^2\vartheta\sin\varphi$$

$$\frac{\partial(z,x)}{\partial(\varphi,\rho)} = \begin{vmatrix} 0 & \cos\vartheta \\ -\rho\sin\vartheta\sin\varphi & \bullet \end{vmatrix} = +\rho\sin\vartheta\cos\vartheta\sin\varphi$$

$$\left[y\, dz\wedge dx = \right. \quad \rho\sin\vartheta\sin\varphi \Big[\rho\cos\varphi \, d\rho\wedge d\vartheta + \rho^2\sin^2\vartheta\sin\varphi \, d\vartheta\wedge d\varphi + \rho\sin\vartheta\cos\vartheta\sin\varphi \, d\varphi\wedge d\rho \Big]$$

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(\rho,\vartheta)}\, d\rho \wedge d\vartheta + \frac{\partial(x,y)}{\partial(\vartheta,\varphi)}\, d\vartheta \wedge d\varphi + \frac{\partial(x,y)}{\partial(\varphi,\rho)}\, d\varphi \wedge d\rho$$

$$\frac{\partial(x,y)}{\partial(\rho,\vartheta)} = \begin{vmatrix} \sin\vartheta\cos\varphi & \rho\cos\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi & \rho\cos\vartheta\sin\varphi \end{vmatrix} = 0$$

$$\frac{\partial(x,y)}{\partial(\vartheta,\varphi)} = \begin{vmatrix} \rho\cos\vartheta\cos\varphi & -\rho\sin\vartheta\sin\varphi \\ \rho\cos\vartheta\sin\varphi & \rho\sin\vartheta\cos\varphi \end{vmatrix} = \rho^2\sin\vartheta\cos\vartheta\cos^2\varphi + \rho^2\sin\vartheta\cos\vartheta\sin^2\varphi = \rho^2\sin\vartheta\cos\vartheta$$

$$\frac{\partial(x,y)}{\partial(\varphi,\rho)} = \begin{vmatrix} -\rho\sin\vartheta\sin\varphi & \sin\vartheta\cos\varphi \\ \rho\sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi \end{vmatrix} = -\rho\sin^2\vartheta\sin^2\varphi - \rho\sin^2\vartheta\cos^2\varphi = -\rho\sin^2\vartheta$$

$$z\, dx \wedge dy = \rho\cos\vartheta\left(\rho^2\sin\vartheta\cos\vartheta\, d\vartheta\wedge d\varphi - \rho\sin^2\vartheta\, d\varphi\wedge d\rho\right)$$

$$x\, dy\wedge dz + \ldots = \overbrace{\left(-\rho^2\sin^2\vartheta\cos\varphi\sin\varphi + \rho^2\sin^2\vartheta\sin\varphi\cos\varphi\right)}^{0} d\rho\wedge d\vartheta +$$

$$\underbrace{\left(\rho^2\sin^2\vartheta\cos\vartheta\cos^2\varphi + \rho^2\sin^2\vartheta\cos\vartheta\sin^2\varphi - \rho^2\sin^2\vartheta\cos\vartheta\right)}_{= 0} d\varphi\wedge d\rho +$$

$$\left(+\rho^3\sin^3\vartheta\cos^2\varphi + \rho^3\sin^3\vartheta\sin^2\varphi + \rho^3\cos^2\vartheta\sin\vartheta\right) d\vartheta\wedge d\varphi$$

$$+\rho^3\left[\sin^3\vartheta + \cos^2\vartheta\sin\vartheta\right] = \boxed{+\rho^3\sin\vartheta\, d\vartheta\wedge d\varphi}$$

$$\Rightarrow \boxed{f^*E = + \sin\vartheta \, d\vartheta \wedge d\varphi} = d(-\cos\vartheta) \wedge d\varphi$$

$$\int_0^{\pi}\int_0^{2\pi} (+\sin\vartheta \, d\vartheta \wedge d\varphi) = -2\pi \int_0^{\pi} d\cos\vartheta \; d$$

$$= -2\pi \cos\vartheta \Big|_0^{\pi}$$

$$= -2\pi(-1-1)$$

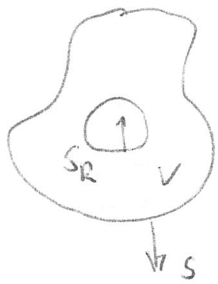
$$= +4\pi$$

f^*E è chiusa: $d(d(-\cos\vartheta)\wedge d\varphi) = 0$

$\Rightarrow E$ è chiusa (f è diffeom.)

$$f^* dE = d f^*E = 0$$

$$\int_{\substack{S \setminus S_R \\ \| \\ \partial V}} E = \int_V dE = 0$$



f^*E non è esatta: se lo fosse: $f^*E = d\alpha$

$$\int_{S_R} d\alpha = \int_{\partial S = 0} \alpha = 0 \qquad \text{mentre} \quad \int_{S_R} f^*E = 4\pi$$

Anche E non è esatta, altrimenti se $E = d\alpha$

$$f^*E = f^* d\alpha = d f^*\alpha$$

(Ricordare: f diffeo $\displaystyle\int_{f(S)} \omega = \int_S f^*\omega$)

cambiamento di variabili