TOPOLOGIA E GEOMETRIA DIFFERENZIALE

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(1) In $\mathbb{R}^3$, si consideri

$$\omega = x\, dy \wedge dz - y\, dz \wedge dx$$

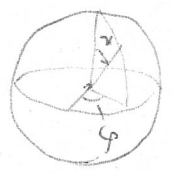
Si dica se ω è chiusa. È esatta? In caso affermativo si determini $\chi_0 \in \Lambda^1$ tale che $\omega = d\chi_0$. Si determinino tutte le $\chi \in \Lambda^1$ tali che $\omega = d\chi$.

(2) In $(\mathbb{R}^2, \omega = dq \wedge dp)$ si consideri

$$X = \sin p \frac{\partial}{\partial q} + \alpha(p) \frac{\partial}{\partial p} \qquad \alpha \in C^\infty$$

Si determinino le α per le quali X risulti simplettico. In tal caso X è hamiltoniano? In caso affermativo, si determini un'hamiltoniana di X, e se ne descrivano le curve integrali.

(3) In $(S^2, g = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$



Sia $X = \frac{\partial}{\partial \varphi}$. Si determini $X^\flat$ e si calcoli $\mathcal{L}_X X^\flat$. Date poi $f_1 = \sin\vartheta$, $f_2 = \sin\varphi$, si trovino ∇f_1 e ∇f_2 (∇: gradiente relativo a g)

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate

(1) $\omega = x\, dy \wedge dz - y\, dz \wedge dx$

$$d\omega = dx \wedge dy \wedge dz - dy \wedge dz \wedge dx$$
$$= dx \wedge dy \wedge dz - dx \wedge dy \wedge dz = 0$$

ω è esatta: $\omega = d\chi$ (Poincaré)

$$\chi = \alpha\, dx + \beta\, dy + \gamma\, dz$$
$$d\chi = \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx \wedge dy + \left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right) dy \wedge dz + \left(\frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}\right) dz \wedge dx$$

$$\begin{cases} \dfrac{\partial\alpha}{\partial x} - \dfrac{\partial\beta}{\partial y} = 0 \\ \dfrac{\partial\gamma}{\partial y} - \dfrac{\partial\beta}{\partial z} = x \\ \dfrac{\partial\alpha}{\partial z} - \dfrac{\partial\gamma}{\partial x} = -y \end{cases}$$

$\beta = 0$

$\gamma = xy$

$\alpha = 0$

$\frac{\partial\gamma}{\partial y} = x$

$\frac{\partial\gamma}{\partial x} = y$ ✓

$$\omega = d(\overbrace{xy\, dz}^{\chi_0})$$

Check: $w = (dx\, y + x\, dy)\, dz =$
$$= y\, dx \wedge dz + x\, dy \wedge dz$$
$$= -y\, dz \wedge dx + x\, dy \wedge dz$$

Sol: $\chi = \chi_0 + df$ (Poincaré)

(2) $X = \sin p \dfrac{\partial}{\partial q} + \alpha(p) \dfrac{\partial}{\partial p}$ $\qquad \omega = dq \wedge dp$

X hamiltoniano

$$i_X \omega = i_X (dq \wedge dp) = i_X dq \wedge dp - i_X dp \wedge dq$$

$$= \overbrace{dq(X)}^{X(q)}\, dp - \overbrace{dp(X)}^{X(p)}\, dq$$

$$= \sin p\, dp - \alpha(p)\, dq$$

$$= -\alpha(p)\, dq + \sin p\, dp$$

$d(i_X \omega) = 0 \qquad \dfrac{\partial \sin p}{\partial q} + \dfrac{\partial \alpha}{\partial p} = 0$

$$\Rightarrow \frac{\partial \alpha}{\partial p} = 0 \Rightarrow \alpha = c$$

$$\Rightarrow X = \sin p \frac{\partial}{\partial q} + c \frac{\partial}{\partial p}$$

controllo, in altro modo: $\mathcal{L}_X \omega = \mathcal{L}_X dq \wedge dp + dq \wedge \mathcal{L}_X dp$

$$= d\mathcal{L}_X(q) \wedge dp + dq \wedge d\mathcal{L}_X(p)$$

$$= \underbrace{d(\sin p) \wedge dp}_{\overset{\shortparallel}{0}} + \underbrace{dq \wedge dc}_{\overset{\shortparallel}{0}} = 0 \quad \checkmark$$

X è hamiltoniano (poincaré)

$$i_X \omega = -c\, dq + \sin p\, dp = d(\overbrace{-cq - \cos p}^{\lambda_X} + \underbrace{\xi}_{\text{cost}})$$

Curve integrali $\lambda_X = \text{cost}$

(3) $M = S^2 \qquad g = d\vartheta^2 + \sin^2\vartheta \, d\varphi^2$

$$\boxed{\begin{aligned} v^i &= g^{ij} v_j \\ v_i &= g_{ij} v^j \end{aligned}}$$

$$X = \frac{\partial}{\partial\varphi} \qquad X^\flat \doteq \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin^2\vartheta \end{pmatrix}$$

$$X^\flat = \sin^2\vartheta \, d\varphi$$

$$\mathcal{L}_X X^\flat = \mathcal{L}_{\frac{\partial}{\partial\varphi}} (\sin^2\vartheta \, d\varphi) = \underbrace{\mathcal{L}_{\frac{\partial}{\partial\varphi}} (\sin^2\vartheta)}_{=0} d\varphi + \sin^2\vartheta \underbrace{\mathcal{L}_{\frac{\partial}{\partial\varphi}} d\varphi}_{=0}$$

$$= 0$$

$$= 0$$

$$f_1 = \sin\vartheta \qquad df_1 = \cos\vartheta \, d\vartheta$$

$$\nabla f_1 = (df_1)^\sharp \qquad \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\vartheta} \end{pmatrix} \begin{pmatrix} \cos\vartheta \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta \\ 0 \end{pmatrix}$$

$$\boxed{\nabla f_1 = \cos\vartheta \frac{\partial}{\partial\theta}}$$

$$f_2 = \sin\varphi \qquad df_2 = \cos\varphi \, d\varphi$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\vartheta} \end{pmatrix} \begin{pmatrix} 0 \\ \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\cos\varphi}{\sin^2\vartheta} \end{pmatrix}$$

$$\boxed{\nabla f_2 = \frac{\cos\varphi}{\sin^2\vartheta} \frac{\partial}{\partial\varphi}}$$