TOPOLOGIA E GEOMETRIA DIFFERENZIALE
vecchio programma

Prova scritta del

18 gennaio 2019

(1) in $\mathbb{R}^3$ siano $X = x\frac{\partial}{\partial x}$, $T = x\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial y}\otimes dz$

Calcolare $\mathcal{L}_X T$

(2) in $\{x>0, y>0\}$ sia data la metrica
$$g = x^2 dx^2 + dy^2$$
e $X_\alpha = x^\alpha \frac{\partial}{\partial x}$. Determinare α in modo che X_α sia di Killing per g.

(3) In $\{x>0, y>0\} \subset \mathbb{R}^3$ siano dati
$$X_\alpha = x^\alpha\frac{\partial}{\partial x} \quad , \quad Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$
Calcolare $[X_\alpha, Y]$ e determinare α in modo che $[X_\alpha, Y] = 0$. Detta $\Delta = \langle X, Y\rangle$ in $\mathbb{R}^3$ (distr. individuata da X, Y) ($X = X_\alpha$ per il quale $[X,Y]=0$)

trovarne le sottovarietà integrali in $\mathbb{R}^3$

Tempo a disposizione 1h

Le risposte vanno adeguatamente giustificate

Topogeo
(vecchio programma)

① $X = x \frac{\partial}{\partial x}$

$T = x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes dz$

$$\mathcal{L}_X T = \mathcal{L}_X(x) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes dz + x\, \mathcal{L}_X\left(\frac{\partial}{\partial x}\right) \otimes \frac{\partial}{\partial y} \otimes dz + x \frac{\partial}{\partial x} \otimes \mathcal{L}_X\left(\frac{\partial}{\partial y}\right) \otimes dz + x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes \mathcal{L}_X dz$$

$$\mathcal{L}_X(x) = X(x) = x \underbrace{\frac{\partial x}{\partial x}}_{1} = x$$

$$\mathcal{L}_X \frac{\partial}{\partial x} = \left[x \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial x}$$

$$\mathcal{L}_X \frac{\partial}{\partial y} = \left[x \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$$

$$\mathcal{L}_X dz = d\, \mathcal{L}_X z = d\left(x \frac{\partial z}{\partial x}\right) = d\,0 = 0$$

$$= x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes dz - x \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \otimes dz + 0 + 0 = 0$$

$$\boxed{\mathcal{L}_X T = 0}$$

in $\{ x>0, y>0 \}$

② $g = x^2 dx^2 + dy^2$

$X = x^\alpha \frac{\partial}{\partial x}$ Det. α in modo che X sia di Killing

$$\mathcal{L}_X g = \mathcal{L}_X (x^2 dx^2) \cdot dy^2 + x^2 dx^2 \cdot \underbrace{\mathcal{L}_X dy^2}_{= 0}$$

$$\mathcal{L}_X dy^2 = \mathcal{L}_X dy \, dy + dy \, \mathcal{L}_X dy$$

$$\mathcal{L}_X dy = d \mathcal{L}_X y = 0$$

$$\mathcal{L}_X (x^2 dx^2) = (\mathcal{L}_X x^2) dx^2 + x^2 \mathcal{L}_X dx^2$$

$$= x^\alpha \underbrace{\frac{\partial x^2}{\partial x}}_{2x} dx^2 + x^2 \Big(\underbrace{\mathcal{L}_X (dx)}_{=\, d\mathcal{L}_X x} dx + dx \, \mathcal{L}_X (dx) \Big)$$

$$d\mathcal{L}_X x = d[X(x)] = d\left[x^\alpha \frac{\partial}{\partial x} x\right] = d x^\alpha = \alpha x^{\alpha - 1} dx$$

$$= 2 x^{\alpha+1} dx^2 + 2 x^2 \alpha x^{\alpha-1} dx^2$$

$$= 2 x^{\alpha+1} [1 + \alpha] dx^2$$

$\Rightarrow$ X killing $\Leftrightarrow$ $\alpha = -1$

$x > 0$
$y > 0$

③

$$X = x^\alpha \frac{\partial}{\partial x}$$

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$[X, Y] = \left[x^\alpha \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] = -\frac{\partial x^\alpha}{\partial x}\frac{\partial}{\partial x}$$

$$= \alpha x^{\alpha - 1} \cdot \frac{\partial}{\partial x}$$

$$[X, Y] = 0 \iff \alpha = 0$$

$\rightsquigarrow$ $X = \frac{\partial}{\partial x}$

$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

sottovarietà integrali

$$\langle X, Y \rangle = \left\langle \frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \right\rangle$$

$$\omega = \alpha\, dx + \beta\, dy + \gamma\, dz$$

$$\omega\left(\frac{\partial}{\partial x}\right) = 0 \longrightarrow \alpha = 0$$

$$\omega\left(y\frac{\partial}{\partial y}\right) = 0 \qquad \beta y = 0 \implies \beta = 0$$

$$\omega = \gamma\, dz \qquad \gamma \neq 0$$

$\implies$ $\rightsquigarrow$ sott. var. integrali $z = c$ (piani)