TOPOLOGIA E GEOMETRIA DIFFERENZIALE vecchio programma

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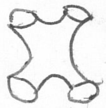
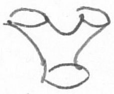
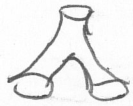
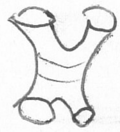
Prova Scritta del 13/7/2018

(1) Calcolare $H^*\Big($  $\Big)$ in due modi

(Supponendo nota $H^*\big($  $\big)$)

1: Applicando Mayer-Vietoris nel modo seguente

$U =$  $V =$  $U \cup V = M =$ 

$U \cap V =$ 

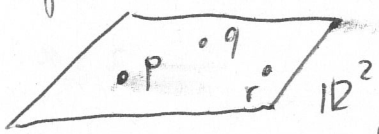
2. Applicando M-V nel modo seguente

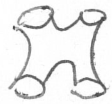
$U = \mathbb{R}^2 \setminus \{p, q\}$

$V = \mathbb{R}^2 \setminus \{r\}$

$U \cup V = \mathbb{R}^2$

$U \cap V = M = \mathbb{R}^2 \setminus \{p, q, r\}$



$\mathbb{R}^2 \simeq$ 

(poli!)

(2) Sia $T = e^x \frac{\partial}{\partial x} \otimes dx \otimes dz$ $\quad X = yz \frac{\partial}{\partial x}$ (in $\mathbb{R}^3$)

Calcolare $\mathcal{L}_X T$

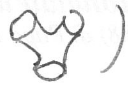
(3) in $\mathbb{R}^3$, sia $\omega = dx \wedge dy \wedge dz$, $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha z\frac{\partial}{\partial z}$ $\quad \alpha \in \mathbb{R}$

Determinare $\alpha \in \mathbb{R}$ in modo che $\mathcal{L}_X \omega = 0$

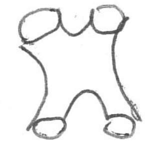
Quale proprietà ha il flusso $\{F_t^X\}$?

Tempo a disposizione: 2h

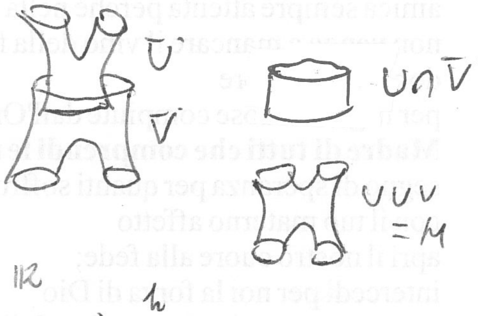
Le risposte vanno adeguatamente giustificate

① Supponendo noto che $H^*\Big($  $\Big) = \begin{cases} \mathbb{R} & q=0 \\ \mathbb{R}^2 & q=1 \\ 0 & q=2 \end{cases}$

$\cong \mathbb{R}^2 \setminus \{p,q\}$

1 Determinare $H^*\Big($  $\Big)$ tramite Mayer-Vietoris

applicato nel modo seguente



$$0 \dashrightarrow \overset{\mathbb{R}}{H^0(M)} \xrightarrow{f} \overset{\mathbb{R}}{H^0(\underset{t}{U})} \oplus \overset{\mathbb{R}}{H^0(\underset{w}{V})} \xrightarrow{g} \overset{\mathbb{R}}{H^0(\underset{w-t}{U\cap V})} \xrightarrow{h}$$

$$\xrightarrow{h} \boxed{H^1(M)} \xrightarrow{i} \overset{\mathbb{R}^2}{H^1(\underset{w}{U})} \oplus \overset{\mathbb{R}^2}{H^1(\underset{t}{V})} \xrightarrow{j} \overset{\mathbb{R}}{H^1(\underset{t-w}{U\cap V})} \xrightarrow{k} (0)$$

$$\xrightarrow{\mathbb{R}} \boxed{H^2(M)} \to \underset{0}{H^2(U)} \oplus \underset{0}{H^2(V)} \to \underset{0}{H^2(U\cap V)} \to 0$$

$H^2(M) = 0$

M non compatta

$$h^1 = \dim \operatorname{Ker} i + \dim \operatorname{Im} \overset{3}{i}$$

$$= \dim \underbrace{\operatorname{Im} h}_{0} + \dim \underbrace{\operatorname{Im} i}_{3}$$

$\operatorname{Ker} k = \mathbb{R} = \operatorname{Im} j \quad \dim(\operatorname{Ker} j) = 4 - \dim(\operatorname{Im} j) = 4-1 = 3$

$\operatorname{Ker} j = \dim \operatorname{Im} i \qquad \dim \operatorname{Im} i = 3$

$\dim \operatorname{Im} h = 1 - \dim \operatorname{Ker} h = 1 - \dim \underbrace{(\operatorname{Im} g)}_{=1} = 1 - 1 = 0$

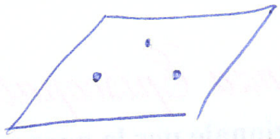
$\Rightarrow h^1 = 3$

Variante: $1 - 2 + 1 - \overset{h^1}{x} + \overset{3}{\widetilde{4 - 1}} = 0$

$\boxed{x = 3}$

$H^*\Big($  $\Big) = \begin{cases} \mathbb{R} & q=0 \\ \mathbb{R}^3 & q=1 \\ 0 & q=2 \end{cases}$

Calcoliamo anche così: $M = \mathbb{R}^2 \setminus \{p, q, r\}$



$$U = \mathbb{R}^2 \setminus \{p, q\}$$
$$V = \mathbb{R}^2 \setminus \{r\}$$
$$U \cup V = \mathbb{R}^2$$
$$U \cap V = M$$

MV:

$$0 \to \overset{\mathbb{R}}{H^0(\mathbb{R}^2)} \to \overset{\mathbb{R}}{H^0(U)} \oplus \overset{\mathbb{R}}{H^0(V)} \to \overset{\mathbb{R}}{H^0(M)} \to$$
$$\overset{0}{H^1(\mathbb{R}^2)} \to \overset{\mathbb{R}^2}{H^1(U)} \oplus \overset{\mathbb{R}}{H^1(V)} \to \boxed{H^1(M)} \to$$
$$\overset{0}{H^2(\mathbb{R}^2)} \to \underset{0}{H^2(U)} \oplus \underset{0}{H^2(V)} \to \underset{\overset{\|}{0}}{H^2(M)} \to 0$$

Det. $H^1(M)$: $\quad 0 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} H^1(M) \xrightarrow{h} 0$

$$h^1 = \dim \operatorname{Ker} h = \dim \operatorname{Im} g$$

anche dir: g è un isomorfismo $\Rightarrow h^1 = 3$

$$3 = \dim \operatorname{Im} g + \dim \operatorname{Ker} g$$
$$= \dim \operatorname{Im} g + \underbrace{\dim \operatorname{Im} f}_{= 0}$$

$$\Rightarrow h^1 = 3 \qquad (h^0 = 1,\ h^2 = 0)$$

(variante:

$$3 - h^1 = 0 \quad)$$

(2) $T = e^{x} \frac{\partial}{\partial x} \otimes dx \otimes dz$

$X = zy \frac{\partial}{\partial x}$

$$\mathcal{L}_X T = (\mathcal{L}_X e^{x}) \frac{\partial}{\partial x} \otimes dx \otimes dz + e^{x} (\mathcal{L}_X \frac{\partial}{\partial x}) \otimes dx \otimes dz$$
$$+ e^{x} \frac{\partial}{\partial x} \otimes (\mathcal{L}_X dx) \otimes dz$$
$$+ e^{x} \frac{\partial}{\partial x} \otimes dx \otimes \mathcal{L}_X dz$$

- $\mathcal{L}_X e^{x} = X(e^{x}) = zy \frac{\partial e^{x}}{\partial x} = zy e^{x}$
- $\mathcal{L}_X \frac{\partial}{\partial x} = \left[zy \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right] = 0$
- $\mathcal{L}_X dx = d \mathcal{L}_X x = d X(x) = d(zy \frac{\partial x}{\partial x}) = d(zy)$
  $= y dz + z dy$
- $\mathcal{L}_X dz = d \mathcal{L}_X z = d X(z) = d \cdot 0 = 0$

$$\boxed{\mathcal{L}_X T = zy e^{x} \frac{\partial}{\partial x} \otimes dx \otimes dz + e^{x} y \frac{\partial}{\partial x} dz \otimes dz + e^{x} z \frac{\partial}{\partial x} \otimes dy \otimes dz}$$

(3) $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha z \frac{\partial}{\partial z}$

trovare $\alpha \in \mathbb{R}$ in modo che $\mathcal{L}_X (dx\wedge dy \wedge dz) = 0$

1. $\mathcal{L}_X(\underbrace{dx\wedge dy\wedge dz}_{\omega}) \overset{\varepsilon}{=} \operatorname{div} X \; dx\wedge dy\wedge dz$

$$= (1+1+\alpha)\, dx\wedge dy\wedge dz$$
$$= (2+\alpha)\, dx\wedge dy\wedge dz$$

$\mathcal{L}_X \omega = 0 \iff \alpha = -2$

2. Facciamo il conto direttamente

$$\mathcal{L}_X \omega = \mathcal{L}_X dx\wedge dy\wedge dz + dx\wedge \mathcal{L}_X dy \wedge dz + dx\wedge dy\wedge \mathcal{L}_X dz$$

$\mathcal{L}_X dx = d(\mathcal{L}_X x) = dx$

$\mathcal{L}_X dy = d\mathcal{L}_X y = dy$

$\mathcal{L}_X dz = d\mathcal{L}_X z = \alpha\, dz$

$\mathcal{L}_X \omega = dx\wedge dy\wedge dz$
$+ dx\wedge dy\wedge dz$
$+ \alpha\, dx\wedge dy \wedge dz$
$= (2+\alpha)\, dx\wedge dy\wedge dz$

$= 0 \iff \alpha = -2$

$\{F_t^X\}$ conserva il volume